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The flow of a tubular film. Part 1. Formal mathematical representation

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An expansion scheme is developed to describe the steady axisymmetric flow of a thin tubular liquid film of varying radius; the necessary small parameter is provided by the ratio between the characteristic film thickness and the characteristic tube radius. The co-ordinate system used is an orthogonal one based on the fluid interface and the fluid streamlines. The differential equations that arise thus treat the metric as an unknown set of variables. The method is restricted to situations dominated by viscous forces. Reference is made to numerical solutions that have been obtained in connexion with an industrial polymer-film-blowing process.

1. Introduction

The practical problem which motivated the development of the formal expansions outlined below concerns the steady axisymmetric flow of a thin tubular liquid film of varying radius. This flow is an idealization of part of the process of film-blowing which is used to manufacture thin films of polymeric materials (e.g. polyethylene). A sketch of the process (figure 1), shows the region of interest which lies between the die exit where the free-surface flow begins and the 'freeze-line' where the polymer solidifies. Part 2 of this paper contains a more detailed description of the process (Pearson & Petrie 1969).

The work discussed in this part of the paper provides a formal basis for part 2 where equivalent equations are derived by more direct physical and geometrical arguments, and solutions are presented in order to suggest the practical relevance of the work. The more formal approach presented here gives a better picture of the approximations that are introduced, though of course the magnitude of their effects can only be assessed by a comparison of numerical predictions with experimental results. The approach also provides a basis for the incorporation of several of the neglected factors into the analysis in order to improve the mathematical model of the process.

One of the principal difficulties in this problem is connected with the fact that the position of the boundaries to the fluid flow is not known *a priori*, not even approximately. This difficulty is met here by choosing a co-ordinate system,

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which has to be determined during the solution of the problem, such that the free surface and the boundary conditions there are expressible in simple terms, at the expense of more complicated equations describing the flow as a whole (and more complicated initial, i.e. upstream, conditions). Even this does not lead to a manageable problem without taking further steps.

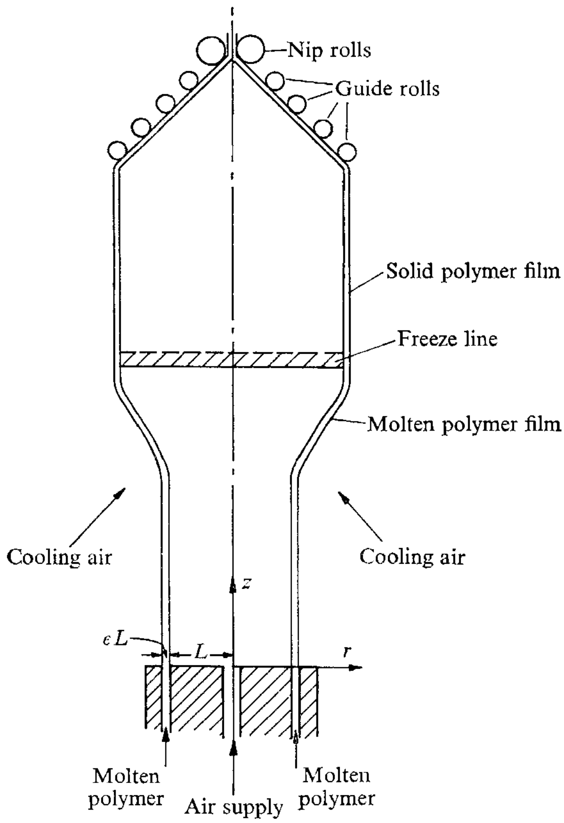

FIGURE 1. Sketch of the film-blowing process. Up to the guide rolls there is symmetry about the axis $r = 0$. The rolls are all cylinders with axes normal to the plane drawn. The guide rolls collapse the film from a tubular shape and the nip rolls seal the top edge of the bubble. Air is supplied only to maintain a pressure P above atmospheric pressure inside the bubble.

If we restrict consideration to an orthogonal co-ordinate system and to steady axisymmetric flow (without swirl) the equations become manageable, since the streamlines may be used as one set of co-ordinate lines. Thus we obtain the three-dimensional equivalent of the intrinsic equations of motion for two-dimensional flow (cf. Milne-Thomson 1955, p. 570). The other co-ordinate lines are the envelopes of the normals and binormals to the streamlines, the latter coinciding with the vortex lines for the axisymmetric flows considered here. The approach leads to a natural co-ordinate system in which to perform the expansions set out below for a thin film, and the condition of thinness is readily expressed in terms of the scale factors associated with the co-ordinate system. This type of approach has also been used by Pearson (1967, p. 73) in a perturbation approach to the lubrication approximation.

2. Formulation of the problem

For the steady flow the equations of motion and continuity may be written in general (curvilinear) co-ordinates using dimensionless variables as

$$R\sum_{j=1}^{3} u^j u^i_{,j} = \sum_{j=1}^{3}\left\{-g^{ij}\frac{\partial p}{\partial x^j}+\sigma^{ij}_{,j}\right\}, \quad \text{for} \quad i = 1,2,3 \tag{1}$$

and

$$\sum_{j=1}^{3} u^i_{,i} = 0. \tag{2}$$

The notation used is standard (covariant and contravariant) tensor notation with commas denoting covariant differentiation with respect to the co-ordinates x^1, x^2 and x^3, except that the summation convention is not used, i.e. all sums are explicitly stated. R is the Reynolds number, $\rho UL/\mu$, where ρ is the fluid density (the fluid is taken to be incompressible), μ the fluid viscosity (or some 'typical' viscosity for a non-Newtonian fluid), U a typical fluid velocity and L a typical length. (The choice of U and L is discussed below.) Physical quantities (denoted by a tilde) are used to define the dimensionless components of the velocity vector, u^i, the extra-stress tensor, σ^{ij}, the metric tensor, g^{ij}, and the pressure, p. Thus, using bracketed suffixes to denote physical (rather than co- or contra-variant) components of velocity and stress

$$\tilde{u}_{(i)} = U(\tilde{g}_{ii})^{\frac{1}{2}}L^{-1}u^i,$$

$$\tilde{g}_{ij} = L^2 g_{ij},$$

$$\tilde{\sigma}_{(i)(j)} = \mu U(\tilde{g}_{ii}\tilde{g}_{jj})^{\frac{1}{2}}L^{-3}\sigma^{ij},$$

$$\tilde{p} = \mu UL^{-1}p,$$

and the total stress has components $(-\tilde{p}\tilde{g}_{ij}+\tilde{\sigma}_{ij})$.

Now for a Newtonian fluid the extra-stress is given by

$$\tilde{\sigma}_{(i)(j)} = 2\mu\tilde{e}_{(i)(j)} = \mu(\tilde{u}_{(i),(j)}+\tilde{u}_{(j),(i)})$$

and for an orthogonal co-ordinate system the metric tensor may be written

$$g_{ij} = \begin{pmatrix} h_1^2 & 0 & 0 \\ 0 & h_2^2 & 0 \\ 0 & 0 & h_3^2 \end{pmatrix}.$$

If the streamlines are used as the x^1 co-ordinate lines (so that $u^i = (u^1, 0, 0)$) and the condition of axisymmetry (and no swirl) is used to remove partial derivatives with respect to x^3, the rate of strain tensor has only five non-zero components,

$$\left.\begin{aligned}
&e_1^1 = u^1 \frac{\partial}{\partial x^1} \ln (u^1 h_1), \\
&e_2^2 = u^1 \frac{\partial \ln h_2}{\partial x^1}, \qquad e_3^3 = u^1 \frac{\partial \ln h_3}{\partial x^1}, \\
&e_2^1 = \frac{1}{2} \frac{\partial u^1}{\partial x^2}, \qquad e_1^2 = \frac{1}{2} \frac{h_1^2}{h_2^2} \frac{\partial u^1}{\partial x^2}
\end{aligned}\right\} \tag{3}$$

(and in dimensionless form $\sigma_j^i = 2e_j^i$). [Compare Batchelor (1967, p. 600), where physical components $e_{(i)(j)}$ and $u_{(i)}$ ($\equiv u^i h_i$) are used.]

Using these results and the compatibility relations for the scale factors h_i (discussed below) equations (1) and (2) reduce to

$$\frac{\partial p}{\partial x^1} - R(u^1 h_1)^2 \frac{\partial \ln h_2 h_3}{\partial x^1} = \frac{h_1}{h_2 h_3} \frac{\partial}{\partial x^2} \left\{ \frac{h_3}{h_1 h_2} \frac{\partial (u^1 h_1^2)}{\partial x^2} \right\}, \tag{4}$$

$$\frac{\partial p}{\partial x^2} - R(u^1 h_1)^2 \frac{\partial \ln h_1}{\partial x^2} = -u^1 h_2^2 \frac{\partial}{\partial x^1} \left\{ \frac{1}{u^1 h_1^2 h_2^2} \frac{\partial (u^1 h_1^2)}{\partial x^2} \right\} \tag{5}$$

and

$$\frac{\partial}{\partial x^1} \ln (u^1 h_1 h_2 h_3) = 0. \tag{6}$$

The compatibility relations ensure that the three functions h_1^2, h_2^2 and h_3^2 are the non-zero covariant components of a diagonal metric tensor in Euclidean space, and that therefore the order of covariant differentiation is immaterial (given continuity of the appropriate partial derivatives). The conventional way of deriving these relations (see, for example, McConnell 1957, p. 152 ff.) is from the condition that the Riemann–Christoffel tensor is identically zero and there are in general six such relations (also called the Lamé relations). For orthogonal co-ordinates these are given in McConnell (1957, p. 156, as example 9(vi)), and here there is some redundancy. When the condition of axisymmetry (in the form of independence of quantities from x^3) is used, the four equations below relating the three (non-zero) scale factors are obtained.

$$\left.\begin{aligned}
\frac{\partial^2 h_3}{\partial x^{1^2}} &= \frac{\partial h_3}{\partial x^1} \frac{\partial \ln h_1}{\partial x^1} - \frac{h_1^2}{h_2^2} \frac{\partial h_3}{\partial x^2} \frac{\partial \ln h_1}{\partial x^2}, \\
\frac{\partial^2 h_3}{\partial x^{2^2}} &= \frac{\partial h_3}{\partial x^2} \frac{\partial \ln h_2}{\partial x^2} - \frac{h_2^2}{h_1^2} \frac{\partial h_3}{\partial x^1} \frac{\partial \ln h_2}{\partial x^1}, \\
\frac{\partial^2 h_3}{\partial x^1 \partial x^2} &= \frac{\partial h_3}{\partial x^1} \frac{\partial \ln h_1}{\partial x^2} + \frac{\partial h_3}{\partial x^2} \frac{\partial \ln h_2}{\partial x^1}, \\
h_2 \frac{\partial^2 h_2}{\partial x^{1^2}} + h_1 \frac{\partial^2 h_1}{\partial x^{2^2}} &= \frac{h_1}{h_2} \frac{\partial h_1}{\partial x^2} \frac{\partial h_2}{\partial x^2} + \frac{h_2}{h_1} \frac{\partial h_1}{\partial x^1} \frac{\partial h_2}{\partial x^1}.
\end{aligned}\right\} \tag{7a–d}$$

Here also there is a degree of redundancy and it may be shown that it is sufficient to satisfy two of the first three equations everywhere and the third along some axisymmetric surface (which is for (7*a*) any surface x^2 = constant, or for (7*b*) any surface x^1 = constant) and it follows that all four equations are satisfied.

However, for convenience all four equations are retained and employed later, while realizing that essentially there are five independent equations relating the five dependent variables, p, u^1, h_1, h_2 and h_3.

Note. There is an analogous situation in linear elasticity where six strain components must satisfy three equilibrium equations and six compatibility relations. Here the difficulty may be avoided by going back to the underlying problem involving three displacement components which are given by the three equilibrium equations. A similar approach here would be possible, with three equations (continuity and two momentum equations) and the three basic unknowns being pressure and the co-ordinates (r, z of the cylindrical polar co-ordinates) of a material particle. The authors are grateful to Professor A. E. Green for pointing out the analogy.

For a discussion of the general ideas underlying this part of the work see Aris (1962).

3. Boundary conditions

Before discussing boundary conditions, we must be more specific about the choice of co-ordinate system. It will help to introduce cylindrical polar co-ordinates (r, ψ, z) (with r the distance from the axis of symmetry, z the distance along the axis of symmetry, and ψ the angular co-ordinate, see figure 2). We may clearly choose $x^3 \equiv \psi$, in which case the scale factor h_3 becomes r (and independence from x^3 does in fact mean axisymmetry). The choice of the streamlines as the x^1 co-ordinate lines means that stream tubes are surfaces x^2 = constant

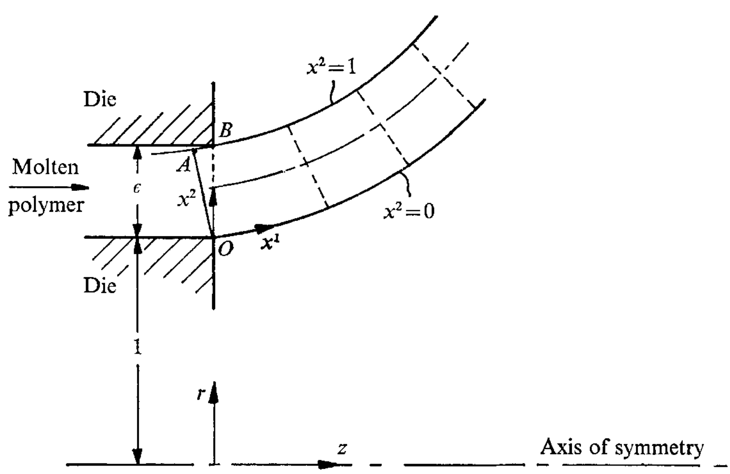

FIGURE 2. Idealized die exit geometry. (Lengths shown are dimensionless.)

and we may choose $x^2 = 0$ and $x^2 = 1$ for the two free surfaces of the film. The origin is chosen at the inner edge of the die exit, so that $x^1 = 0, x^2 = 0$ corresponds to $z = 0$, $r = 1$. (This implies the choice of the length scale L to be the initial internal radius of the bubble.) If the dimensionless thickness of the film is ϵ, then the outer edge of the die exit is $z = 0$, $r = 1+\epsilon$ or $x^2 = 1$, $x^1 =$ constant. [The constant will only be zero if the streamlines at the die exit are all parallel to the axis of symmetry which need not be the case.] It is now still possible to specify the scale factors h_1 and h_2, each along one co-ordinate line, since physical distances correspond to quantities $h_i dx^i$ (see, for example, Darboux 1910, p. 164). For h_2 we specify that on $x^1 = 0$, h_2 is constant, and the condition that the surface $x^2 = 1$ passes through the points $r = 1+\epsilon$, $z = 0$ will suffice to determine the constant (since the die gap, ϵ, is also given by the integral $\int_0^1 h_2(\partial r/\partial x^2)_z \,\partial x^2$ evaluated at $z = 0$, and the geometry of the streamlines determines everything there except h_2). We may specify h_1 along $x^2 = 0$ and for simplicity have chosen $h_1 = 1$.

The free surface boundary conditions, of continuity of stress, lead to two equations at $x^2 = 0$ and two at $x^2 = 1$. These are, at the inner surface ($x^2 = 0$), that

$$\left.\begin{aligned} p - P + \frac{\Gamma}{h_2}\left(\frac{1}{h_1}\frac{\partial h_1}{\partial x^2} + \frac{1}{h_3}\frac{\partial h_3}{\partial x^2}\right) &= \sigma^2_2 - P^2_2 \\ \text{and} \qquad \sigma^1_2 - P^1_2 + \frac{h_2}{h_1^2}\frac{\partial \Gamma}{\partial x^1} &= 0, \end{aligned}\right\} \tag{8}$$

where P is the constant (dimensionless) excess (i.e. above atmospheric) pressure inside the bubble, Γ is the dimensionless surface tension, $\tilde{\Gamma}/\mu U$ (or the ratio of Reynolds number to Weber number $Wb = \rho U^2 L/\tilde{\Gamma}$ and $\tilde{\Gamma}$ is the physical surface tension), P^i_j is the dimensionless viscous stress tensor for the air, representing the effect of air drag. At the outer surface, $x^2 = 1$, we have

$$\left.\begin{aligned} p - \frac{\Gamma}{h_2}\left(\frac{1}{h_1}\frac{\partial h_1}{\partial x^2} + \frac{1}{h_3}\frac{\partial h_3}{\partial x^2}\right) &= \sigma^2_2 - P^2_2 \\ \text{and} \qquad \sigma^1_2 - P^1_2 - \frac{h_2}{h_1^2}\frac{\partial \Gamma}{\partial x^1} &= 0. \end{aligned}\right\} \tag{9}$$

When we use (3) to replace σ^i_j and neglect surface tension and air drag (see discussion) we obtain:

$$\text{at } x^2 = 0 \qquad \left.\begin{aligned} p - P &= 2u^1 \frac{\partial \ln h_2}{\partial x^1}, \\ \frac{\partial u^1}{\partial x^2} &= 0; \end{aligned}\right\} \tag{10}$$

$$\text{and at } x^2 = 1 \qquad \left.\begin{aligned} p &= 2u^1 \frac{\partial \ln h_2}{\partial x^1}, \\ \frac{\partial u^1}{\partial x^2} &= 0. \end{aligned}\right\} \tag{11}$$

The details of the upstream (initial) and downstream conditions are best considered in connexion with the asymptotic series expansion discussed next, since they are most naturally specified for $z =$ constant and need to be transferred to $x^1 =$ constant. The velocity scale U is chosen to make a mean flow rate unity, and this again is left until the most convenient choice is obvious.

4. Asymptotic series approximation to the problem

The general problem contained in (4)–(7) is hardly tractable, and so an approximation for a thin film is sought. The film thickness is $\int_0^1 h_2\,dx^2$, and so is $O(h_2)$ on the length scale of the bubble radius (L). The die gap gives the initial film thickness to be ϵL, so that the scale factor h_2 is $O(\epsilon)$, while the other scale factors are $O(1)$. Therefore if we postulate that the dependent variables are expressible asymptotically (as $\epsilon \to 0$) as power series in ϵ, these will have the form

$$\left.\begin{aligned} h_1 &= h_{01} + \epsilon h_{11} + \epsilon^2 h_{21} + \dots,\\ h_2 &= \qquad\ \ \epsilon h_{12} + \epsilon^2 h_{22} + \dots,\\ h_3 &= h_{03} + \epsilon h_{13} + \epsilon^2 h_{23} + \dots,\\ u^1 &= u_0 + \epsilon u_1 + \epsilon^2 u_2 + \dots,\\ p &= p_0 + \epsilon p_1 + \epsilon^2 p_2 + \dots,\\ \theta &= \theta_0 + \epsilon\theta_1 + \epsilon^2\theta_2 + \dots,\\ P &= P_0 + \epsilon P_1 + \epsilon^2 P_2 + \dots. \end{aligned}\right\} \tag{12}$$

Here θ is the angle between a streamline at any point and the direction of the axis of symmetry, which is used later. u^1 is $O(1)$ by choice of velocity scale and p (measured relative to atmospheric pressure) may be shown to be $O(1)$ (not $O(\epsilon^{-1})$ as in the lubrication approximation because of the free surface boundary conditions applicable here).

Formal substitution of the series (12) into equations (4)–(7) and boundary conditions (10) and (11) followed by the equating of powers of ϵ leads to the equations of appendix A. Those that are used in the solution of the problem are set out, with some simplifications (e.g. the obvious use of a first-order equation to simplify a second-order one).

The process of reduction of the 17 equations and 14 boundary conditions of appendix A to a pair of equations in the film thickness (h_{12}) and bubble radius (h_{03}) is somewhat involved and will not be presented in detail. The stages will be outlined briefly and then some alternative approaches which throw light on some of the unexpected features of the derivation will be discussed. In particular, an attempt is made to explain the appearance of terms of higher order (e.g. h_{11}, h_{21}) in the equations which must be considered to obtain a complete 'first-order' problem.

The first-order equations (A 1), (A 4), (A 6), (A 8), (A 11), (A 13), (A 15), together with boundary conditions on u_0 (A 20) and on h_{12} (A 26), show that the five 'first-order' variables u_0, p_0, h_{01}, h_{12}, h_{03}, are independent of x^2, and these equations are then all satisfied identically except for continuity (A 6) in which ϕ_0 must be constant. The further first-order boundary conditions (A 18), (A 19). (A 25) give equations for p_0 and h_{01} and require that P_0 is zero (discussed later), We are therefore left with three equations relating these five functions of x^1, namely

$$u_0 h_{01} h_{12} h_{03} = \phi_0 \text{ (constant)}, \tag{13}$$

$$h_{01} = 1, \tag{14}$$

and

$$p_0 = \frac{2u_0}{h_{12}} \frac{\partial h_{12}}{\partial x^1}. \tag{15}$$

These may be thought of as giving h_{01}, u_0 and p_0 in terms of h_{12} and h_{03}.

These results lead to a considerable simplification of the second-order equations (A 2), (A 5), (A 7), (A 9), (A 12), (A 14), (A 16) and from these (and boundary conditions (A 23) and (A 28)) it is deduced that the second-order variables are linear in x^2. The relevant part of the second-order problem is then rewritten in terms of the five functions of x^1, $\partial u_1/\partial x^2$, $\partial p_1/\partial x^2$, $\partial h_{11}/\partial x^2$, $\partial h_{22}/\partial x^2$ and $\partial h_{13}/\partial x^2$; and we obtain, using boundary conditions (A 21) and (A 22), five more equations involving these five new functions.

We get a soluble problem by taking the third-order equations (A 3), (A 10) and (A 17) and boundary condition (A 24), since these introduce only the two new functions of x^1, $\partial^2 h_{21}/\partial x^{2^2}$ and $\partial^2 u_2/\partial x^{2^2}$. (In fact the velocity derivatives $\partial u_1/\partial x^2$ and $\partial u_2/\partial x^2$ are zero everywhere.) The elimination of the higher order quantities to get two equations in h_{03} and h_{12} is algebraic apart from one integration to obtain $\partial h_{13}/\partial x^2$, and we get

$$\frac{P_1}{2\phi_0} h_{03}\sqrt{(1-h_{03}'^2)} = h_{03}'' \left(\frac{2h_{12}'}{h_{12}} + \frac{h_{03}'}{h_{03}}\right) + \frac{(1-h_{03}'^2)}{h_{03}} \left(\frac{h_{03}'}{h_{03}} - \frac{h_{12}'}{h_{12}}\right), \tag{16}$$

and

$$2\left(\frac{h_{12}''}{h_{12}} - \frac{h_{12}'^2}{h_{12}^2}\right) - \frac{h_{12}'}{h_{12}}\frac{h_{03}'}{h_{03}} + \frac{h_{03}''}{h_{03}} = 0, \tag{17}$$

where $h' \equiv dh/dx^1$, and the Reynolds number has been put equal to zero. (More details of this work are set out in appendix B.)

There now follow the outlines of alternative approaches to two aspects of this formulation, the geometrical aspect and the series expansion aspect in connexion with the x^2 dependence of quantities. Then an attempt is made to throw some light on the necessity of the appearance of 'first-', 'second-' and 'third-order' quantities in obtaining a soluble 'first-order' problem.

An approach to the geometrical part of the problem, which replaces the use of the compatibility relations, is to introduce the angle $\theta(x^1, x^2)$ between a streamline (at a point) and the direction of the axis of symmetry. Then the relationship between the (x^1, x^2, x^3) co-ordinate system and cylindrical polar co-ordinates (r, ψ, z) is summarized by the equations

$$\left.\begin{aligned} x^3 &= \psi, \\ h_3 &= r, \\ dz &= h_1 \cos\theta\, dx^1 - h_2 \sin\theta\, dx^2, \\ dr &= h_1 \sin\theta\, dx^1 + h_2 \cos\theta\, dx^2. \end{aligned}\right\} \tag{18a–d}$$

From these we extract the relevant equations:

$$\frac{\partial}{\partial x^2}(h_1 \cos\theta) = \frac{\partial}{\partial x^1}(-h_2 \sin\theta),$$

$$\frac{\partial h_3}{\partial x^2} = h_2 \cos\theta,$$

$$\frac{\partial h_3}{\partial x^1} = h_1 \sin\theta,$$

and from the latter two

$$\frac{\partial}{\partial x^2}(h_1 \sin\theta) = \frac{\partial}{\partial x^1}(h_2 \cos\theta).$$

Putting in the postulated asymptotic series (12) for the dependent variables gives the following (equating powers of ϵ):

$$\frac{\partial}{\partial x^2}(h_{01} \cos\theta_0) = 0,$$

$$\frac{\partial}{\partial x^2}(h_{01} \sin\theta_0) = 0,$$

$$\frac{\partial h_{03}}{\partial x^2} = 0,$$

$$\frac{\partial h_{03}}{\partial x^1} = h_{01} \sin\theta_0 \quad \text{(an extra equation for the extra variable, } \theta_0\text{)},$$

and the first-order equations of continuity and momentum show that the quantities u_0, h_{12} and p_0 are also independent of x^2. Also we have

$$\frac{\partial h_{13}}{\partial x^2} = h_{12} \cos\theta_0,$$

$$\cos\theta_0 \frac{\partial h_{11}}{\partial x^2} - h_{01} \sin\theta_0 \frac{\partial \theta_1}{\partial x^2} = -\frac{\partial h_{12}}{\partial x^1} \sin\theta_0 - h_{12} \cos\theta_0 \frac{\partial \theta_0}{\partial x^1},$$

$$\sin\theta_0 \frac{\partial h_{11}}{\partial x^2} + h_{01} \cos\theta_0 \frac{\partial \theta_1}{\partial x^2} = \frac{\partial h_{12}}{\partial x^1} \cos\theta_0 - h_{12} \sin\theta_0 \frac{\partial \theta_0}{\partial x^1};$$

three equations giving $\partial h_{13}/\partial x^2$, $\partial h_{11}/\partial x^2$ and $\partial \theta_1/\partial x^2$ (as functions of x^1 only), and we may get two third-order equations relating only $\partial^2 h_{21}/\partial x^{2^2}$, $\partial^2 \theta_2/\partial x^{2^2}$ and known quantities.

Thus we obtain the same information as from the four compatibility relations (7), by a process which has some advantages in case of geometrical interpretation but which perhaps lends itself less easily to use in more general situations.

A reformulation of the series expansion stage of the process avoids the tedious deductions as to the x^2 dependence of the various quantities, and also leads to the use of only one equation from most of the original equations, (4)–(7). This procedure does seem to have disadvantages (in dealing with boundary conditions and in an apparant loss of generality) and is most convincingly justified by the method used above.

The reformulation is carried out as follows: change variable from x^2 to y, $\equiv \epsilon x^2$, and write ϵH_2 for h_2. This merely leads to the formal replacement of x^2 by y and h_2 by H_2 in equations (4)–(7). Now the free surfaces are $y = 0$ and $y = \epsilon$, and if the variables are expressed as Taylor series about $y = 0$ with the obvious equivalence with the quantities defined in equation (12):

$$[\text{E.g. } u^1 = u^1(x^1, 0) + y\frac{\partial u^1}{\partial y} + \frac{y^2}{2}\frac{\partial^2 u^1}{\partial y^2} + \dots,$$

$$= u_0(x^1) + y\frac{\partial u_1}{\partial x^2} + \frac{y^2}{2}\frac{\partial^2 u_2}{\partial x^{2^2}}; \text{ the derivatives are functions of } x^1 \text{ (evaluated at } x^2 = 0).]$$

We get by neglecting $O(y)$ terms, equations derivable from (A 3), (A 5), (A 6), (A 10), (A 14) and (A 17). [These come from equations (4), (5), (6), (7*a*), (7*c*) and (7*d*); from (7*b*) we get an equation for $\partial^2 h_{23}/\partial x^{2^2}$ in terms of 'first-' and 'second-order' quantities.]

Neither of these alternatives assist in providing an explanation of the occurrence of 'first-', 'second-' and 'third-order' quantities, e.g. h_{01}, h_{11} and h_{21}, which all are required in deriving the final pair of equations governing h_{03} and h_{12}, the bubble radius and film thickness.† This explanation may, in part at least, be obtained by considering the interrelation of the important physical effects, namely the viscous stresses due to the thinning of the film, the geometrical effect of the curvature of the film, and the pressure inside the bubble. The principal radii of curvature of the film are of order h_{03} and $d^2h_{03}/\partial x^{1^2}$ (dh_{03}/dx^1 is also involved—see part 2) and the viscous stresses are of order $(1/h_{12})\, dh_{12}/dx^1$ and $(1/h_{03})\, dh_{03}/dx^1$. The internal pressure must balance forces of order ϵh_{12} times these stresses divided by the radii of curvature, and so is $O(\epsilon)$. [A higher pressure would correspond to a situation where the assumptions implicit in using and differentiating the expansions (12) were not valid, and would not permit the physical situation considered.]

The non-zero d^2h_{03}/dx^{1^2} implies a variation in scale factor h_1 of order ϵ across the film, so that $\epsilon \partial h_{11}/\partial x^2$ is closely related to d^2h_{03}/dx^{1^2}. Further, the thinning of the film necessary to give rise to the viscous stresses implies a convergence of the streamlines and consequently a curvature of the x^2 co-ordinate lines (orthogonal to the streamlines). This curvature involves $(1/h_2^2)\, \partial^2 h_1/\partial x^{2^2}$ and is of order 1, so that $\partial^2 h_1/\partial x^{2^2}$ is of order ϵ^2; i.e. $\partial^2 h_{21}/\partial x^{2^2}$ is the relevant quantity and this is closely bound up with the flow pattern giving rise to the viscous stresses. Similarly, it would be possible to indicate, on this somewhat intuitive basis, why all the other quantities involved are relevant in the first-order problem as it has been posed.

† The authors are grateful to one of the referees for pointing out the similar situation which occurs in the theory of a solitary wave in shallow water (cf. Keller 1948, p. 335 in particular).

5. Initial conditions

The initial conditions for the problem are most naturally specified at the die exit, that is at $z = 0$, while the problem has been posed in terms of x^1. Therefore it is necessary to transfer the conditions from $z = 0$ to $x^1 = 0$, using equations (18) and the expansions (12). Figure 2 illustrates the geometry of the die exit and shows the two sets of co-ordinate lines there. ($z = 0$ on OB, $x^1 = 0$ on OA.)

At constant z we see from (18*c*) that

$$h_1 \cos\theta \, dx^1 - h_2 \sin\theta \, dx^2 = 0,$$

so that, using this and (18*d*)

$$dr = h_2 \sec\theta \, dx^2.$$

Using (12) in this
$$dr = \epsilon h_{12} \sec\theta_0 \, dx^2 + O(\epsilon^2),$$

and since h_{12} and θ_0 are independent of x^2 this may be integrated across the film at $z = 0$ to give

$$\epsilon = \epsilon h_{12} \sec\theta_0 |_{x^1=0},$$

and so at $x^1 = 0$
$$h_{12} = \cos\theta_0 = \sqrt{\{1 - (dh_{03}/dx^1)^2\}}.$$

We also know that $h_{03}(0) = 1$, so that for the two second-order ordinary differential equations (16) and (17) we require two more boundary conditions.

Formally these could be the values of dh_{03}/dx^1 and dh_{12}/dx^1 at $x^1 = 0$ (or at some other value of x^1) or the values of h_{03} and h_{12} at some value of x^1 other than zero. The choice relevant to the physical problem which motivated this work is discussed in part 2 of this paper. Reference to equations (3) and (6) shows that the values of dh_{03}/dx^1 and dh_{12}/dx^1 determine the first-order principal rates of strain and hence the principal stresses, so that prescribed surface tractions at a flow boundary correspond to prescribed values of these derivatives.

6. Discussion

The problem tackled is a complicated one, and in this initial attempt at a solution many factors have been ignored, either tacitly or with some explanation. The results which are obtained in part 2 are physically reasonable, which provides some basis for the belief that no effect of major importance has been ignored. The main neglected factors are here discussed briefly and the feasibility of their formal incorporation into the solution scheme is considered. A more physically motivated discussion will be postponed to part 2 of this paper.

Two effects which in practice do appear to be important are those of air drag and of temperature variation. The latter affects the flow through the temperature dependence of the viscosity, so that the energy equation must be introduced, with appropriate boundary conditions, in order to solve the full problem. Since the last stage of the solution is being done numerically, this does not present any difficulty in principle, but the details do not appear to be completely straightforward. In particular, work of Martin (Holmes-Walker & Martin 1966) suggests that the dominating effect is radiative cooling, and the appropriate physical parameters (especially emissivity) are not well known. Formally, an asymptotic series, $T = T_0 + \epsilon T_1 + \epsilon^2 T_2 + \ldots$, for the temperature ($T$) is introduced, and the dependence of μ on T leads to the dependence of P_1 in equation (16) on $T_0(x^1)$, because of the way P was made dimensionless. T_0 is shown to be independent of x^2, and its dependence on x^1 determined from the energy equation and its associated boundary conditions. In part 2 a qualitative idea of the effect of this on the bubble shape is obtained by letting P_1 vary with x^1, in a predetermined manner, simulating the (qualitatively) known dependence of viscosity on x^1.

The air drag, which occasions the terms P^1_2 and P^2_2 in equations (8) and (9), does not appear to allow so straightforward a treatment. An approach to a similar problem is given by Taylor (1959), giving a way of attempting to refine the solutions obtained above which it is more appropriate to discuss in part 2. A formal incorporation of air drag into the work outlined above makes the problem much harder by introducing x^2 dependence of quantities earlier than is convenient for the unsophisticated treatment used. To the extent that the bubble shape is unaffected by the asymptotic process, $\epsilon \to 0$, the airflow will be unaltered, so the stresses at the surface of the film will be $O(1)$. (It may be, of course, that this will not lead to a feasible solution, which could be interpreted by saying that as the film gets thinner it is unable to sustain the constant surface tractions applied, and our original assumptions are no longer going to hold.) Then P^1_2 is $O(\epsilon)$, since P^1_2 is h_2/h_1 times the physical component of the shear stress at the surface, and the boundary conditions on $\partial u^1/\partial x^2$ are altered appropriately. The further difficulty arises that, if this is the only modification to our simplified problem, we still have $\partial^2 u_1/\partial x^{2^2} = 0$, so that the surface shear stresses are constrained to be equal, which implies that the air drag force must act in opposite directions on the two surfaces, which is an unwelcome and physically unreasonable constraint. These considerations lead to the hypothesis that transverse velocity gradients will be non-zero only in a layer near each free surface of thickness $o(\epsilon)$. (It will have to be of thickness $O(\epsilon^n)$, for n an integer $\geqslant 2$, to preserve the simple asymptotic scheme used here.) In effect we are led to a double asymptotic expansion in ϵ and in some dimensionless measure of the air drag. This idea has not been followed up, since it will be extremely complicated, and it is argued that the approach employed here has provided a justification for solutions obtained in regions away from the free surfaces, and that the less formal approach discussed in part 2 is more likely to lead to a prediction of the overall effect of air drag.

There appears to be no difficulty in principle in incorporating the effects of inertia and gravity (neither of which are believed to be important here). The assumption once again is that these factors are unaffected by the asymptotic process, so that the Reynolds and Froude numbers are both $O(1)$. (The process is one of letting the film thickness tend to zero while keeping its superficial velocity, not the volumetric flow rate, constant.) The effect of inertia is then found by retaining the term in R in each of equations (A 3), (A 5) and (A 4) (where it vanishes since $\partial h_{01}/\partial x^2 = 0$) and we merely have to solve (numerically) equations differing slightly from (16) and (17) with one extra parameter, R, to specify for each computation.

The appropriate terms to insert into (4) and (5) to take account of gravity are, when gravity acts parallel to the axis of symmetry in the direction of z decreasing, $G(h_1/h_2)\,\partial h_3/\partial x^2$ to be added to $\partial p/\partial x^1$ and $-G(h_2/h_1)\,\partial h_3/\partial x^1$ to be added to $\partial p/\partial x^2$. Here $G = gL^2/\nu U$ or $G = R/F$, where F is the Froude number U^2/Lg. These terms will then contribute to equations (A 2), (A 3) and (A 5) when $G = O(1)$, though the contribution to (A 2) will be zero since $\partial h_{03}/\partial x^2 = 0$ still holds. Terms $G(h_{01}/h_{12})\,\partial h_{13}/\partial x^2$ and $-G(h_{12}/h_{01})\,\partial h_{03}/\partial x^1$ are thus added to $\partial p_0/\partial x^1$ in (A 3) and $\partial p_1/\partial x^2$ in (A 5) respectively.

If we wish to try to take account of surface tension we must retain the terms in Γ from equations (8) and (9). Since Γ is $O(1)$, being independent of the asymptotic limiting process, the significant equations for the first-order problem are different. The equation which corresponds to (16) (which is essentially a balance between viscous forces and a pressure difference of order ϵ) is a balance between an order 1 pressure difference and surface tension forces. Thus the asymptotic process provides no justification for the neglect of surface tension, and indeed it must give rise to the dominant forces for very thin films. This point has not been followed up in detail, and formally the relevance of the work reported here could be doubted. Physical reasoning does suggest that if we consider a particular film thickness the viscous forces may be an order of magnitude greater than the surface tension forces while the film is thin enough for the geometrical approximations to give a reasonably accurate picture and physically useful predictions.

Two assumptions which were important in obtaining manageable equations were those of steadiness and of axisymmetry. It appears that unsteady flow would be difficult to treat by the method used here. A time-varying co-ordinate system would involve the sort of complications that arise in a Lagrangian formulation of the equations of motion (cf. Lodge 1964, p. 328, and Batchelor 1967, p. 71). It is felt that for non-steady flows in general other approaches may be more fruitful. The loss of axisymmetry would also lead to a more complicated problem, and the asymptotic expansion will lead to partial differential equations in x^1 and x^3 relating the variable here found to depend on x^1 alone.

One case where it might be possible to adopt this approach to take account of these factors is that which will arise in a linearized stability analysis of the flow found here. It seems reasonable to suppose that there will be no excessive difficulty in treating small departures from steadiness and axisymmetry. It seems to the authors that one of the more fruitful lines to pursue following this work will be a study of the feasibility of using this method to determine the stability of the steady axisymmetric flow.

One important aspect of the problem which has been ignored concerns the details of the flow at the upstream and downstream ends of the regions for which a solution has been sought. Dependent to some extent on this, since more upstream conditions (and possibly downstream ones) are required, are the problems of taking account of higher order terms in the asymptotic expansions and of allowing for the elastico-viscous nature of the fluid. [No extra initial conditions appear to be needed for time-independent non-Newtonian behaviour, but for a fluid with memory the history of its flow before entering the region studied must in some sense be given, either by the flow field upstream of $z = 0$ or of derivatives with respect to z of the velocity at $z = 0$.]

In order to incorporate more terms of the asymptotic expansions more details of the velocity at $z = 0$ are required, to provide initial conditions on u_1. It is not possible to set $u_1 = 0$ at $z = 0$ since the requirement that u_0 is independent of x^2 restricts the initial conditions, and in general the initial velocity must be $u_0(0)+O(\epsilon)$, a constant term together with smaller terms allowing a variation of u^1 with x^2. Since in the body of the flow $\partial u_1/\partial x^2$ is zero, these more general initial conditions must be accommodated by means of a transition region where the flow inside the die and the free-surface flow considered here may be matched. This is a singular perturbation problem of some difficulty, which is probably best attacked by a co-ordinate perturbation. It will provide an interesting problem, which the authors feel is best left until apparently more straightforward problems of this nature have been more fully explored (cf. Clarke 1968).

A similar problem arises in connexion with the downstream boundary of the flow field, i.e. the region above the freeze-line (figure 1) where the solidification of the fluid takes place, followed by the stretching, flattening and winding up of the film. This is in some ways a harder problem practically, because of the rapid change of physical properties, but seems less important to further progress with the overall problem, and has not the fundamental significance of the die exit problem.

In order that these difficulties may be avoided, it is argued that the effects of the regions where these transitions occur (both liquid to solid and constrained to free-surface flow transitions) are confined to a neighbourhood of the region concerned and their overall effect may be allowed for by experimentally determined correction factors. Thus the initial values of the bubble radius and film thickness used in computations will not be the actual die dimensions, but will involve, for example, a multiplicative correction to the die gap to allow for the die-swell phenomenon (cf. Pearson 1966, p. 48).

The problem of dealing with the non-Newtonian nature of the fluid—both in its shear-dependent viscosity and in its elastic properties—has not been considered. A generalized power law model for variable viscosity could be used without much additional complication, and effects would be similar to those of viscosity variation with temperature. It would be valuable to treat elastic effects, since here we have an elongational flow as opposed to a viscometric flow, so that material functions other than the three viscometric functions of Coleman & Noll (1961) will be appropriate, and a new class of data for distinguishing constitutive equations would in principle be obtainable. Some models of fluids with memory appear particularly well-suited to the treatment of this paper, e.g. the integral models of Walters (1962) and of Lodge (1964, p. 103). However, the questions of whether the results will be sufficiently sensitive to choice of constitutive equations to be useful in distinguishing experimentally between the many popular alternatives, and of whether useful progress can be made without more success with the die exit problem, remain to be settled.

Against this exposure of a multitude of shortcomings can be set the fact that physically realistic results are obtained. The main features of the observed flow are shown by the numerical results discussed in part 2, and a more detailed criticism, giving indications of the weaker points of the above work, must await comparisons of these numerical results with experimental results. The value of any quantitative predictions based on this work thus remains to be established, but the qualitative picture is one in which the authors have some confidence.

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Appendix A

Formal substitution of the expansions (12) into equations (4), (5), (6) and (7) and into the boundary conditions (10) and (11), followed by equating powers of ϵ leads to the following equations (some simplification of the higher order equations has been carried out):

$$(4)\quad \epsilon^{-2}:\quad \frac{\partial}{\partial x^2}\left(\frac{h_{03}}{h_{01}h_{12}}\frac{\partial u_0 h_{01}^2}{\partial x^2}\right) = 0, \tag{A 1}$$

$$\epsilon^{-1}:\quad \frac{\partial}{\partial x^2}\left\{\frac{h_{03}}{h_{01}h_{12}}\frac{\partial}{\partial x^2}(u_1 h_{01}^2 + 2u_0 h_{01} h_{11}) + \frac{h_{03}}{h_{01}h_{12}}\left(\frac{h_{13}}{h_{03}} - \frac{h_{11}}{h_{01}} - \frac{h_{22}}{h_{12}}\right)\frac{\partial u_0 h_{01}^2}{\partial x^2}\right\} = 0, \tag{A 2}$$

$$\begin{aligned}\epsilon^{0}:\quad \frac{\partial p_0}{\partial x^1} - R\left\{\frac{u_0^2 h_{01}^2}{h_{12}h_{03}}\frac{\partial h_{12}h_{03}}{\partial x^1}\right\} = \frac{h_{01}}{h_{12}h_{03}}\frac{\partial}{\partial x^2}\Bigg[\frac{h_{03}}{h_{01}h_{12}} &\times\left\{\frac{\partial}{\partial x^2}(u_2 h_{01}^2 + 2u_1 h_{01}h_{11} + 2u_0 h_{01}h_{21} + u_0 h_{11}^2) + \left(\frac{h_{13}}{h_{03}} - \frac{h_{11}}{h_{01}} - \frac{h_{22}}{h_{12}}\right)\right. \\ &\times \frac{\partial}{\partial x^2}(u_1 h_{01}^2 + 2u_0 h_{01}h_{11}) + \left(\frac{h_{23}}{h_{03}} - \frac{h_{21}}{h_{01}} - \frac{h_{32}}{h_{12}} + \frac{h_{11}^2}{h_{01}^2} + \frac{2h_{11}h_{22}}{h_{01}h_{12}}\right. \\ &\left.\left. + \frac{h_{22}^2}{h_{12}^2} - \frac{h_{13}h_{11}}{h_{03}h_{01}} - \frac{h_{13}h_{22}}{h_{02}h_{12}}\right)\frac{\partial u_0 h_{01}^2}{\partial x^2}\right\}\Bigg].\end{aligned} \tag{A 3}$$

$$(5)\quad \epsilon^{0}:\quad \frac{\partial p_0}{\partial x^2} - R\left\{u_0^2 h_{01}\frac{\partial h_{01}}{\partial x^2}\right\} = -u_0 h_{12}^2 \frac{\partial}{\partial x^1}\left\{\frac{1}{u_0 h_{01}^2 h_{12}^2}\frac{\partial u_0 h_{01}^2}{\partial x^2}\right\}, \tag{A 4}$$

$$\begin{aligned}\epsilon^{1}:\quad &\frac{\partial p_1}{\partial x^2} - R\left\{u_0^2 h_{01}\frac{\partial h_{11}}{\partial x^2} + (u_0^2 h_{11} + 2u_0 u_1 h_{01})\frac{\partial h_{01}}{\partial x^2}\right\} \\ &= -(u_1 h_{12}^2 + 2u_0 h_{12}h_{22})\frac{\partial}{\partial x^1}\left\{\frac{1}{u_0 h_{01}^2 h_{12}^2}\frac{\partial u_0 h_{01}^2}{\partial x^2}\right\} \\ &\quad - u_0 h_{12}^2\frac{\partial}{\partial x^1}\left[\frac{1}{u_0 h_{01}^2 h_{12}^2}\left\{\frac{\partial}{\partial x^2}(u_1 h_{01}^2 + 2u_0 h_{01}h_{11}) - \left(\frac{u_1}{u_0} + \frac{2h_{11}}{h_{01}} + \frac{2h_{22}}{h_{12}}\right)\frac{\partial u_0 h_{01}^2}{\partial x^2}\right\}\right].\end{aligned} \tag{A 5}$$

(6) ϵ^0: $u_0 h_{01} h_{12} h_{03} = \phi_0(x^2),$ (A 6)

ϵ^1: $\dfrac{u_1}{u_0}+\dfrac{h_{11}}{h_{01}}+\dfrac{h_{22}}{h_{12}}+\dfrac{h_{13}}{h_{03}} = \phi_1(x^2)$ (A 7)

(after one integration with respect to x^1).

(7a) ϵ^{-2}: $\dfrac{\partial h_{03}}{\partial x^2}\dfrac{\partial h_{01}}{\partial x^2} = 0,$ (A 8)

ϵ^{-1}: $\dfrac{\partial h_{13}}{\partial x^2}\dfrac{\partial h_{01}}{\partial x^2}+\dfrac{\partial h_{03}}{\partial x^2}\dfrac{\partial h_{11}}{\partial x^2} = 0,$ (A 9)

ϵ^0: $\dfrac{\partial}{\partial x^1}\left(\dfrac{1}{h_{01}}\dfrac{\partial h_{03}}{\partial x^1}\right) = -\dfrac{1}{h_{12}^2}\left(\dfrac{\partial h_{23}}{\partial x^2}\dfrac{\partial h_{01}}{\partial x^2}+\dfrac{\partial h_{13}}{\partial x^2}\dfrac{\partial h_{11}}{\partial x^2}+\dfrac{\partial h_{03}}{\partial x^2}\dfrac{\partial h_{21}}{\partial x^2}\right).$ (A 10)

(7b) ϵ^0: $\dfrac{\partial}{\partial x^2}\left(\dfrac{1}{h_{12}}\dfrac{\partial h_{03}}{\partial x^2}\right) = 0,$ (A 11)

ϵ^1: $h_{12}\dfrac{\partial}{\partial x^2}\left(\dfrac{1}{h_{12}}\dfrac{\partial h_{13}}{\partial x^2}\right) = \dfrac{\partial h_{03}}{\partial x^2}\dfrac{\partial (h_{22}/h_{12})}{\partial x^2}.$ (A 12)

(7c) ϵ^0: $h_{12}\dfrac{\partial}{\partial x^1}\left(\dfrac{1}{h_{12}}\dfrac{\partial h_{03}}{\partial x^2}\right) = \dfrac{1}{h_{01}}\dfrac{\partial h_{03}}{\partial x^1}\dfrac{\partial h_{01}}{\partial x^2},$ (A 13)

ϵ^1: $h_{12}\dfrac{\partial}{\partial x^1}\left(\dfrac{1}{h_{12}}\dfrac{\partial h_{13}}{\partial x^2}\right) = \dfrac{\partial h_{03}}{\partial x^2}\dfrac{\partial (h_{22}/h_{12})}{\partial x^2}+\dfrac{1}{h_{01}}\dfrac{\partial h_{13}}{\partial x^1}\dfrac{\partial h_{01}}{\partial x^2}+\dfrac{\partial h_{03}}{\partial x^1}\dfrac{\partial (h_{11}/h_{01})}{\partial x^2}.$ (A 14)

(7d) ϵ^0: $\dfrac{\partial}{\partial x^2}\left(\dfrac{1}{h_{12}}\dfrac{\partial h_{01}}{\partial x^2}\right) = 0,$ (A 15)

ϵ^1: $h_{12}\dfrac{\partial}{\partial x^2}\left(\dfrac{1}{h_{12}}\dfrac{\partial h_{11}}{\partial x^2}\right) = \dfrac{\partial h_{01}}{\partial x^2}\dfrac{\partial (h_{22}/h_{12})}{\partial x^2},$ (A 16)

ϵ^2: $h_{01}h_{12}\left\{\dfrac{\partial}{\partial x^2}\left(\dfrac{1}{h_{12}}\dfrac{\partial h_{21}}{\partial x^2}\right)+\dfrac{\partial}{\partial x^1}\left(\dfrac{1}{h_{01}}\dfrac{\partial h_{12}}{\partial x^1}\right)\right\}$

$= h_{01}\dfrac{\partial h_{11}}{\partial x^2}\dfrac{\partial (h_{22}/h_{12})}{\partial x^2}+h_{01}\dfrac{\partial h_{01}}{\partial x^2}\dfrac{\partial}{\partial x^2}\left(\dfrac{h_{32}}{h_{12}}-\dfrac{h_{22}^2}{2h_{12}^2}\right).$ (A 17)

The free-surface boundary conditions, neglecting the effects of air drag and surface tension, and writing the excess pressure inside the bubble, P, as a series, $P_0+\epsilon P_1+\ldots$, are (for all x^1):

(10a) ϵ^0: $p_0 - P_0 = \dfrac{2u_0}{h_{12}}\dfrac{\partial h_{12}}{\partial x^1}$ at $x^2 = 0,$ (A 18)

(11a) ϵ^0: $p_0 = \dfrac{2u_0}{h_{12}}\dfrac{\partial h_{12}}{\partial x^1}$ at $x^2 = 1,$ (A 19)

(10b, 11b) ϵ^0: $\dfrac{\partial u_0}{\partial x^2} = 0$ at $x^2 = 0, 1.$ (A 20)

(10a) ϵ^1: $p_1 - P_1 = \dfrac{2u_1}{h_{12}}\dfrac{\partial h_{12}}{\partial x^1}+2u_0\dfrac{\partial (h_{22}/h_{12})}{\partial x^1}$ at $x^2 = 0,$ (A 21)

$$(11a)\ \epsilon^1: \quad p_1 = \frac{2u_1}{h_{12}}\frac{\partial h_{12}}{\partial x^1} + 2u_0\frac{\partial(h_{22}/h_{12})}{\partial x^1} \quad \text{at} \quad x^2 = 1, \tag{A 22}$$

$$(10b,\ 11b)\ \epsilon^1: \quad \frac{\partial u_1}{\partial x^2} = 0 \quad \text{at} \quad x^2 = 0, 1. \tag{A 23}$$

$$(10b,\ 11b)\ \epsilon^2: \quad \frac{\partial u_2}{\partial x^2} = 0 \quad \text{at} \quad x^2 = 0, 1. \tag{A 24}$$

The conditions on the scale factors (§ 3) lead to

$$h_{01} = 1 \quad \text{at} \quad x^2 = 0, \text{ all } x^1, \tag{A 25}$$

$$h_{12} \text{ is const.} \quad \text{at} \quad x^1 = 0, \text{ all } x^2, \tag{A 26}$$

$$h_{11} = 0 \quad \text{at} \quad x^2 = 0, \text{ all } x^1, \tag{A 27}$$

$$h_{22} = 0 \quad \text{at} \quad x^1 = 0, \text{ all } x^2. \tag{A 28}$$

Appendix B. Derivation of equations (16) and (17)

Following the scheme outlined in § 4, from the results there obtained that u_0, p_0, h_{12} and h_{03} are independent of x^2, and from equations (13), (14) and (15) (which give h_{01} and u_0 and p_0 in terms of h_{12} and h_{03}) the equations of appendix A are simplified. Equations (A 12) and (A 16) lead to $\partial^2 h_{13}/\partial x^{2^2} = 0$, and $\partial^2 h_{11}/\partial x^{2^2} = 0$, implying that $\partial h_{13}/\partial x^2$ and $\partial h_{11}/\partial x^2$ are independent of x^2. Then equation (A 2), which reduces to $\partial^2(u_1 + 2u_0 h_{11})/\partial x^{2^2} = 0$, implies that $\partial u_1/\partial x_2$ is independent of x^2 and so, from boundary conditions (A 23), is zero everywhere. Equation (A 5) gives

$$\begin{aligned}\frac{\partial p_1}{\partial x^2} &= -u_0 h_{12}^2 \frac{\partial}{\partial x^1}\left\{\frac{1}{u_0 h_{12}^2}\frac{\partial}{\partial x^2}(u_1 + 2u_0 h_{11})\right\}, \\ &= -2u_0 h_{12}^2 \frac{\partial}{\partial x^1}\left(\frac{1}{h_{12}^2}\frac{\partial h_{11}}{\partial x^2}\right),\end{aligned} \tag{B 1}$$

whence $\partial p_1/\partial x^2$ is also a function of x^1 only.

Differentiating equation (A 7) and using the foregoing gives

$$\begin{aligned}\frac{\partial h_{11}}{\partial x^2} + \frac{1}{h_{12}}\frac{\partial h_{22}}{\partial x^2} + \frac{1}{h_{03}}\frac{\partial h_{13}}{\partial x^2} &= \phi_1'(x^2) \\ &= \text{const.}\end{aligned} \tag{B 2}$$

from condition (A 28) and so $\partial h_{22}/\partial x^2$ is independent of x^2. Equations (A 17) and (A 3) respectively give

$$\frac{\partial^2 h_{21}}{\partial x^{2^2}} + h_{12}\frac{\partial^2 h_{12}}{\partial x^{1^2}} = \frac{1}{h_{12}}\frac{\partial h_{11}}{\partial x^2}\frac{\partial h_{22}}{\partial x^2} \tag{B 3}$$

and

$$\begin{aligned}\frac{\partial p_0}{\partial x^1} = \frac{1}{h_{12}^2}\frac{\partial^2}{\partial x^{2^2}}(u_2 + 2u_1 h_{11} + 2u_0 h_{21} + u_0 h_{11}^2) \\ + \frac{1}{h_{12}^2}\frac{\partial}{\partial x^2}(u_1 + 2u_0 h_{11})\frac{\partial}{\partial x^2}\left(\frac{h_{13}}{h_{03}} - h_{11} + \frac{h_{22}}{h_{12}}\right),\end{aligned} \tag{B 4}$$

so that $\partial^2 h_{21}/\partial x^{2^2}$ and hence $\partial^2 u_2/\partial x^{2^2}$ are independent of x^2, and boundary condition (A 24) then implies that $\partial u_2/\partial x^2 = 0$ everywhere. Finally equations (A 10) and (A 14) give us

$$\frac{\partial^2 h_{03}}{\partial x^{1^2}} = -\frac{1}{h_{12}^2}\frac{\partial h_{13}}{\partial x^2}\frac{\partial h_{11}}{\partial x^2} \tag{B 5}$$

and

$$h_{12}\frac{\partial}{\partial x^1}\left(\frac{1}{h_{12}}\frac{\partial h_{13}}{\partial x^2}\right) = \frac{\partial h_{03}}{\partial x^1}\frac{\partial h_{11}}{\partial x^2}. \tag{B 6}$$

Now we have six equations relating the seven functions of x^1 (namely h_{12}, h_{03}, $\partial h_{11}/\partial x^2$, $\partial h_{13}/\partial x^2$, $\partial h_{22}/\partial x^2$, $\partial p_1/\partial x^2$ and $\partial^2 h_{21}/\partial x^{2^2}$), assuming that equations (13), and (15) are used to eliminate u_0 and p_0.

The seventh equation is obtained from boundary conditions (A 21) and (A 22) taken together, which give

$$\frac{\partial p_1}{\partial x^2} + P_1 = 2u_0\frac{\partial}{\partial x^1}\left(\frac{1}{h_{12}}\frac{\partial h_{22}}{\partial x^2}\right). \tag{B 7}$$

The first step in the elimination is to solve equations (B 5) and (B 6) for $\partial h_{11}/\partial x^2$ and $\partial h_{13}/\partial x^2$ which proceeds as follows:

$$\frac{\partial}{\partial x^1}\left(\frac{1}{h_{12}}\frac{\partial h_{13}}{\partial x^2}\right) = \frac{\partial h_{03}}{\partial x^1}\frac{\partial^2 h_{03}}{\partial x^{1^2}}\bigg/\left(-\frac{1}{h_{12}}\frac{\partial h_{13}}{\partial x^2}\right),$$

hence integrating

$$\left(\frac{1}{h_{12}}\frac{\partial h_{13}}{\partial x^2}\right)^2 = k - \left(\frac{\partial h_{03}}{\partial x^1}\right)^2$$

and the constant, k, may be shown to be 1 (cf. the geometrical arguments in the latter part of § 4).

Thus we have

$$\partial h_{13}/\partial x^2 = h_{12}\sqrt{[1-(\partial h_{03}/\partial x^1)^2]} \tag{B 8}$$

and

$$\partial h_{11}/\partial x^2 = -h_{12}(\partial^2 h_{03}/\partial x^{1^2})/\sqrt{[1-(\partial h_{03}/\partial x^1)^2]}, \tag{B 9}$$

and we may substitute, using these and using (B 2) for $\partial h_{22}/\partial x^2$, (B 3) for $\partial^2 h_{21}/\partial x^{2^2}$, (B 1) for $\partial p_1/\partial x^2$, to obtain from (B 4) and (B 7) two equations relating h_{12}, h_{03} and their derivatives. (As stated above, (13) and (15) are used to eliminate u_0 and p_0.) The resulting pair of equations are equations (16) and (17) in the main text.

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